Grading guide, Pricing Financial Assets, August 2021

1. (a) Define a forward contract on a stock.

   (b) Consider a stock that does not pay dividends with price $\mathbf{S}_t$ at time t. Assume a constant continuously compounded interest rate of r. Using an arbitrage argument find the forward price F_t at time t on a forward contract on the stock that matures at time $T > t$.

   (c) Consider the value of a forward contract with forward price K at time $t < T$ and assume that the stock price follows the geometric Brownian motion

$$d\mathbf{S}_t = \mu \mathbf{S}_t dt + \sigma \mathbf{S}_t dz$$

   Using Ito's lemma find the process followed by the value of the forward contract. What will the drift be under the risk neutral measure?

   (d) Assume now that the stock pays a constant continuous dividend rate of δ. What will the forward price be?

Solution:

(a)

Definition 0.1 (Forward Contract)**.** *A **Forward Contract** is an agreement to buy or sell a quantity of an asset of a specified quality and type (called **the underlying asset**) delivered at a specified future time T (**the expiration date**) and at a specified place for an agreed **delivery price** K as measured in some defined numeraire/currency.*

The price is initially (at t) set at a level $K = F(t,T)$, such that the value $V_K(t,T)$ of the contract is 0, i.e. by definition

$$V_{F(t,T)}(t,T) = 0$$

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*The price $F(t,T)$ is called **the forward price** at t for delivery at T.*

(b) The following arbitrage argument can be made: Buy the stock spot financed by a loan of S_t repaid at T. The only cashflow on this will be the repayment $S_t \mathrm{e}^{r(T-t)}$ at T, leaving you with the stock. An alternative is to enter into a forward contract with delivery at T. To eliminate arbitrage it must be that the forward price at t for delivery at T is $F_t = \mathbf{S}_t \mathrm{e}^{r(T-t)}$

(c) For a forward contract with forward price K the value is $V_t = \mathbf{S}_t - K\mathrm{e}^{-r(T-t)}$. Using Ito's lemma you get that the differential form for the value process for the forward:

$$dV_t = [\mu \mathbf{S}_t - rK\mathrm{e}^{-r(T-t)}]dt + \sigma \mathbf{S}_t dz$$

For a currently priced forward (i.e. where K gives the forward contract a zero net present value $K = F_t = \mathrm{e}^{r(T-t)}\mathbf{S}_t$) we see that the drift is 0 under the risk neutral measure (by setting $\mu = r$), while a non-negative market value will grow at a rate r. It will be considered ok just to derive this for the currently priced forward.

(d) In the arbitrage argument above you get too much if you buy the stock paying dividend, but you can repay and reduce the interest payments continuously leaving only $\mathbf{S}_t \mathrm{e}^{(r-\delta)(T-t)}$ to repay at T. Alternatively consider the price of a synthetic stock paying no dividends and compare that to $\mathbf{S}_t$ and go on from there. In either case $F_t = \mathbf{S}_t \mathrm{e}^{(r-\delta)(T-t)}$.

2. Assume that a stock with price $\mathbf{S}_t$ at time t pays no dividends before time $T > t$, and that there is a constant continuously compounded risk free interest rate of r.

Let $c(\mathbf{S}_0, K, T, r)$ and $p(\mathbf{S}_0, K, T, r)$ be the price at time $t = 0$ of a European call and a European put, respectively, on the stock with the same strike K and expiry T.

(a) Use the put-call-parity to find a relationship between the Thetas (Θ) of the call and put. Comment on the result.

(b) Consider two call options with the same strike K but different expiry dates T_1 and T_2. Assume that $r \geq 0$ and use an arbitrage argument to show that the price of the longer call, c_2, is (weakly) greater than the price of the shorter call, c_1.

Solution:

(a) The put-call parity for options with same maturity and same strike price on the same stock paying no dividend states that the price of long call less the short put must be equal to the forward price on the stock with the same delivery prices as the strike and the same maturity, i.e.

$$c(\mathbf{S}_0, K, T, r) - p(\mathbf{S}_0, K, T, r) = \mathbf{S}_0 - \mathsf{e}^{-rT} K$$

The Theta is the partial derivative with respect to calendar time, i.e. the negative partial derivative with respect to maturity T. By taking the partial derivative we find:

$$\Theta_c - \Theta_p = -r\mathsf{e}^{-rT} K$$

This depends on the sign of r. E.g. with a positive rate there is from discounting an implicit advantage for the call option of the later payment of the strike compared to the current date (and a similar disadvantage for the put of the strike received later). The negative sign on the right-hand side reflects that this relative advantage of the call between the two otherwise similar options becomes smaller as the maturity date gets closer. Hull (9ed, p.409) reports this for the special case of the BSM-model.

(b) One may first use the same argument as when showing that an early exercise of an American call under these assumptions is non-optimal. Consider a portfolio long the long call, long K zero coupon bonds maturing at T_2 and short one stock. At time T_1 this portfolio has a value of

$$c_2(T_1) + \mathsf{e}^{-r(T_2 - T_1)} K - \mathbf{S}_{T_1}$$

while at T_2 it will have a value of

$$\begin{aligned} c_2(T_2) + K - \mathbf{S}_{T_2} &= \\ max[\mathbf{S}_{T_2} - K; 0] + K - \mathbf{S}_{T_2} &\geq 0 \end{aligned}$$

So if this has a non-negative value at T_2 the the value found at T_1 must also be non-negative i.e.

$$c_2(T_1) \geq \mathbf{S}_{T_1} - \mathsf{e}^{-r(T_2 - T_1)} K$$

It is also perfectly fine to arrive at this conclusion just comparing the value of the long call at T_1 with the value of a forward with maturity T_2 and delivery price equal to the strike K, noting that the call at T_2 only will be exercised in the states where the forward has a positive value.

Now we consider a portfolio of a long position in the long call and a short position in the short call. At time T_1 first consider the case of $\mathbf{S}_{T_1} > K$. Then

$$c_2(T_1) - c_1(T_1) \geq (\mathbf{S}_{T_1} - \mathsf{e}^{-r(T_2-T_1)}K) - (\mathbf{S}_{T_1} - K) > 0$$

Finally consider the case of $\mathbf{S}_{T_1} \leq K$. Then

$$c_2(T_1) - c_1(T_1) \geq c_2(T_1) - 0 \geq 0$$

We have just show that for a European call the theta is non-positive (when rates are non-negative).

3. In the Cox-Ingersoll-Ross (CIR) Model the (instantaneous) short term interest rate r is described by the process:

$$dr = a(b-r)dt + \sigma\sqrt{r}dz$$

where a,b and σ are constants ($\sigma^2 < 2ab$), and dz a Brownian increment.

(a) What does this model mean for the behaviour of the short term interest rate?

(b) In this model the solution for the price of a zero-coupon bond can be written

$$\mathbf{P}(t,T) = A(t,T)\mathsf{e}^{-B(t,T)r(t)}$$

Derive the duration of the bond.

(c) In some models of the short term interest rate r (e.g. the Hull-White model) the drift rate is made a function of calendar time t. What is the purpose of the extra flexibility compared to the CIR (or Vasicek) type of models?

Solution:

(a) It should be noted that the model implies a mean reversion of the spot rate to a level of b at a rate of a. It should also be explained that the volatility term means that the short term rate cannot go negative. (Hull 9th ed, p.710).

(b) The answer depends on the definition of duration applied. The version in Hull (9th ed., p.712) is

$$D = -\frac{\frac{\partial \mathbf{P}}{\partial r}}{\mathbf{P}} = B(t,T)$$

(c) The time-dependent drift term in the Hull-White model is introduced to be able to incorporate a given, initial term structure, making the values derived from it "arbitrage-free" in relation to the existing securities priced on the current term structure (assuming these to be arbitrage free). This is in contrast to the CIR and Vasicek models of the "equilibrium" or "endogenous"-type that put restrictions on the possible initial term structure (as it is a function of the current spot rate with only few parameters), (Hull 9th ed., section 31.2-31.3).